

A Constructive Finite Field Method for Scattering Points on the Surface of d -Dimensional Spheres

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Abstract

We use solutions to quadratic forms in $d + 1$ variables over finite fields to scatter points on the surface of the unit sphere S^d , $d \geq 1$. Applications are given for spherical t designs and generalized s energies.

1 Introduction

In this exploratory article, we present a constructive method for scattering points on the surface of d dimensional spheres which we believe is new and of interest. Indeed, the problem of uniformly distributing points on spheres is an interesting and difficult problem with vast applications in fields as diverse as crystallography, approximation theory, computational complexity, molecular structure, and electrostatics. We refer the interested reader to [1], [5], [6], [9], [10], [11], [15], [16], [19], [20] and the many references cited therein for a comprehensive account of this fascinating subject.

Let $d \geq 1$ be an integer and S^d the d -dimensional unit sphere given by the set of all real solutions to the equation

$$x_1^2 + \cdots + x_{d+1}^2 = 1. \quad (1.1)$$

In two dimensions ($d = 1$) the problem is easily reduced to uniformly distributing N points on a circle and the vertices of the regular N -gon provide an obvious answer. For $d \geq 2$ the problem becomes much more difficult; in fact, there are numerous criteria for uniformity, resulting in different optimal configurations, see [13].

In this article, we will describe a constructive method to scatter points on the surface of S^d using finite fields. After describing our construction, we will apply it to two measures of uniformity on S^d , namely spherical t -designs and generalized s energy.

Finite Field Construction

For an odd prime p , let F_p denote the finite field of integers modulo p . Consider the quadratic form given by (1.1) over F_p . The number $N = N(d, p)$ of solutions of this form is well known and is given by

$$N(d, p) = \begin{cases} p^d - p^{(d-1)/2} \eta((-1)^{(d+1)/2}) & \text{if } d \text{ is odd} \\ p^d + p^{d/2} \eta((-1)^{d/2}) & \text{if } d \text{ is even} \end{cases} \quad (1.2)$$

where η is the quadratic character defined on F_p by $\eta(0) = 0$, $\eta(a) = 1$ if a is a square in F_p , and $\eta(a) = -1$ if a is a non-square in F_p , see [17, Theorems 6.26 and 6.27].

Given a solution vector

$$X = (x_1, \dots, x_{d+1}), \quad x_i \in F_p, \quad 1 \leq i \leq d+1,$$

we may assume without loss of generality that the points x_i are scaled so that they are centered around the origin and are contained in the set

$$\{-(p-1)/2, \dots, (p-1)/2\}.$$

More precisely, if $x_i \in X$, define

$$w_i = \begin{cases} x_i & \text{if } x_i \in \{0, \dots, (p-1)/2\} \\ x_i - p & \text{if } x_i \in \{(p+1)/2, \dots, p-1\}. \end{cases}$$

Then $w_i \in \{-(p-1)/2, \dots, (p-1)/2\}$ and the scaled vector

$$W = (w_1, \dots, w_{d+1}), \quad 1 \leq i \leq d+1$$

solves (1.1) if and only if X solves (1.1).

Denoting by $\|\cdot\|$ the usual Euclidean metric, we multiply each solution vector W by $\frac{1}{\|W\|}$. Clearly each of these normalized points is now on the surface of the unit sphere S^d . Use of the finite field F_p for larger primes p provides a method to increase the number N of points that are placed on the surface of S^d for any fixed $d \geq 1$. For increasing values of p , we obtain an increasing number $N = O(p^d)$ of points scattered on the surface of the unit sphere S^d ; in particular, as $p \rightarrow \infty$ through all odd primes, it is clear that $N \rightarrow \infty$. For each prime p and integer $d \geq 1$, we will henceforth denote the set of points arising from our finite field construction by $X = X(d, p)$.

Let us now describe the point set X produced by the finite field construction and provide some explicit examples for small values of p and d . In each case, we may start with a well chosen set $V = V(d, p)$ of vectors. Then, in order to construct the full set of points $X(d, p)$, we need to consider all points obtained from V by taking ± 1 times the entry in each coordinate, and by permuting the coordinates of each vector, in all possible ways. For small values of d and p , this construction is summarized in the following table.

d	p	$N(d, p)$	$V(d, p)$
1	3	4	$\{(1, 0)\}$
1	5	4	$\{(1, 0)\}$
1	7	8	$\{(1, 0), \frac{1}{\sqrt{2}}(1, 1)\}$
2	3	6	$\{(1, 0, 0)\}$
2	5	30	$\{(1, 0, 0), \frac{1}{\sqrt{2}}(2, 1, 1)\}$
2	7	42	$\{(1, 0, 0), \frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{22}}(3, 3, 2)\}$

Observe that for $p = 3, 5, 7$ and $d = 1$, our construction gives the optimal solution, namely the vertices of the regular N -gon. This, however, is not the case for $p > 7$.

The remainder of this paper is organized as follows. In Section 2, we will discuss spherical t -designs. More precisely, we will prove that for any $d \geq 2$ and any odd prime p , our set of points forms a spherical 3-design. Moreover, given any odd positive integer k , our set of points forms a spherical design of index k . Section 3 introduces the notion of generalized s energies. Here we numerically compare our results with well known theoretical and numerical bounds for these latter quantities using results of Wagner, Kuijlaars and Saff, and Rakhmanov, Saff, and Zhou, see [24], [25], [16], [19], and [20]. Finally, in Section 4, we briefly discuss a natural extension of our method to finite fields of prime power orders. Appendices A and B contain numerical data to illustrate the effectiveness of our construction. The computer programs used to make these calculations can be obtained from the authors. Throughout, C will denote a positive constant which may take on different values at different times.

2 Spherical t -designs

In this section, we will study how well our points are distributed using spherical t -designs, a notion first introduced by Delsarte, Goethals, and Seidel in [8].

Definition 2.1 A finite set X of points on the d -sphere S^d is a *spherical t -design* or a *spherical design of strength t* , if for every polynomial f of total degree t or less, the average value of f over the whole sphere is equal to the arithmetic average of its values on X . If this only holds for homogeneous polynomials of degree t , then X is called a *spherical design of index t* .

In other words, X is of index t if the Chebyshev-type quadrature formula

$$\frac{1}{\sigma_d(S^d)} \int_{S^d} f(\mathbf{x}) d\sigma_d(\mathbf{x}) \approx \frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x}) \quad (2.1)$$

is exact for all homogeneous polynomials $f(\mathbf{x}) = f(x_0, x_1, \dots, x_d)$ of degree t (σ_d denotes the surface measure on S^d). A set X is a t -design if it is of index k for every $k \leq t$.

The concept of spherical t -designs has been studied extensively from various points of view, including representation theory, combinatorics, and approximation theory. For general references see [3] and [8]. The existence of spherical designs for every t and d and large enough $N = |X|$ was first proved by Seymour and Zaslavsky in 1984 [21].

The first general construction of spherical designs for arbitrary t , d , and large enough N was given independently by Wagner [23] and Bajnok [3], who used $N \geq C(d)t^{O(d^4)}$ and $N \geq C(d)t^{O(d^3)}$ points, respectively. This bound was later reduced to $C(d)t^{d^2/2+d/2}$ by Korevaar and Meyers [15]. They believe that the minimum size of a t -design on S^d is $C(d)t^d$. Because of its theoretical and practical importance, there is a keen interest in finding explicit constructions for spherical t -designs on N points with N relatively small with respect to t and d .

For explicit constructions of spherical designs it is often convenient to use the following equivalent definition, for a proof see [8].

Lemma 2.2 *A finite subset X of S^d is a spherical t -design if and only if for every homogeneous harmonic polynomial f of total degree t or less*

$$\sum_{\mathbf{x} \in X} f(\mathbf{x}) = 0.$$

A polynomial $f(x_0, x_1, \dots, x_d)$ is *harmonic* if it satisfies Laplace's equation $\Delta f = 0$. The set of homogeneous harmonic polynomials of degree k over S^d forms the vector space $Harm_k(S^d)$ with

$$\dim Harm_k(S^d) = \binom{d+k}{k} - \binom{d+k-2}{k-2}.$$

In particular, for small values of k we find that $\Phi_k(S^d)$ forms a basis for $Harm_k(S^d)$ where

$$\begin{aligned} \Phi_1(S^d) &= \{x_i | 0 \leq i \leq d\}, \\ \Phi_2(S^d) &= \{x_i x_j | 0 \leq i < j \leq d\} \cup \{x_i^2 - x_{i+1}^2 | 0 \leq i \leq d-1\}, \\ \Phi_3(S^d) &= \{x_i x_j x_k | 0 \leq i < j < k \leq d\} \cup \{x_i^3 - 3x_i x_j^2 | 0 \leq i \neq j \leq d\}, \text{ and} \\ \Phi_4(S^d) &= \{x_i^3 x_j - x_i x_j^3 | 1 \leq i < j \leq d+1\} \\ &\quad \cup \{x_i^4 - 6x_i^2 x_j^2 + x_j^4 | 1 \leq i < j \leq d+1\} \\ &\quad \cup \{x_i^3 x_j - 3x_i x_j x_k^2 | 1 \leq i < j \leq d+1, \\ &\quad \quad \quad 1 \leq k \leq d+1, i \neq k, j \neq k\} \\ &\quad \cup \{x_i x_j x_k x_l | 1 \leq i < j < k < l \leq d+1\}. \end{aligned}$$

We now prove

Proposition 2.3 *For every odd positive integer k , odd prime p , and dimension $d \geq 1$, $X(d, p)$ is a spherical design of index k . Furthermore, $X(d, p)$ is a spherical 3-design.*

Proof. Let us consider our point set $X = X(d, p)$. Since for any reflection T with respect to the hyperplane $x_i = 0$ ($0 \leq i \leq d+1$) we have $X^T = X$, the equation

$$\sum_{\mathbf{x} \in X} f(\mathbf{x}) = 0$$

will hold for every $f = \prod_{i=0}^{d+1} x_i^{k_i} \in \text{Harm}_k(S^d)$ which has at least one variable x_i with an exponent k_i odd. This observation immediately implies that our point set X is a spherical design of index k for every odd integer k .

Furthermore, our point set X also remains fixed under a reflection U with respect to the hyperplane $x_i = x_j$ ($0 \leq i, j \leq d+1$), thus we also have

$$\sum_{\mathbf{x} \in X} f(\mathbf{x}) = 0$$

for the quadratics of the form $x_i^2 - x_{i+1}^2$ ($0 \leq i \leq d-1$). Therefore X satisfies Lemma 2.2 for $t = 3$ and is thus a 3-design. \square

While our construction does not, in general, give a spherical t -design for $t \geq 4$, it is worth mentioning a few cases when it does. Namely, we prove the following.

Proposition 2.4 *The sets $X(3, 3)$, $X(2, 5)$ and $X(3, 5)$ form spherical 5-designs, and $X(1, 7)$ is a spherical 7-design.*

Proof. As noted before, $X(1, 7)$ is the regular octagon on the circle S^1 , hence a well known 7-design (see [6]). To prove that $X(3, 3)$, $X(2, 5)$ and $X(3, 5)$ are spherical 5-designs, it suffices to verify that they are of index 4; this, together with Proposition 2.3 yields that they form 5-designs. To prove that they are of index 4, one can easily check that

$$\sum_{\mathbf{x} \in X} f(\mathbf{x}) = 0$$

for the polynomials in $\Phi_4(S^d)$. \square

Remark. It is worth noting that the minimum size of a spherical 3-design on S^d is $2d+2$, and this is achieved by the vertices of the generalized regular octahedron. The size of our pointset, $N(d, p)$, as given by equation (1.2), is generally a lot larger than this; however, our construction may prove useful for other purposes.

3 Generalized s energy

Given a set of points $\omega_N := \{x_1, \dots, x_N\}$, $N \geq 1$ on S^d and $s > 0$, we define the s energy associated with ω_N by

$$E_d(s, \omega_N) := \sum_{1 \leq i < j \leq N} \frac{1}{\|x_i - x_j\|^s}. \quad (3.1)$$

Fekete points on S^d , see [16] and the references cited therein, are points that minimize the $s > 0$ energy over all sets of N points. Physically, this represents the energy of N charged particles that repel each other according to Coulomb's law. (See also [19] for a discussion of Elliptic Fekete points: the case $s = 0$). From recent work of [7], it is known that points of minimal energy for $s \leq d$ are well separated in the sense that

$$\lim_{N \rightarrow \infty} \frac{1}{\sigma_d(S^d)} \int_{S^d} f(y) d\sigma_d(y) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{y \in \omega_N} f(y)$$

for all continuous functions f on S^d . Another way to understand this is to define for a given set of points ω_N ,

$$\delta(\omega_N) := \inf_{i \neq j} \|x_{i,N} - x_{j,N}\|; \quad \delta_N := \sup_{\omega_N \subset S^d} \delta(\omega_N).$$

The determination of δ_N is called *Tammes problem* or the *Spherical packing problem*, see [6]. It asks to maximize the smallest distance among N points on S^d . More precisely, in [16], the authors have shown that uniformly for any $s > d \geq 2$,

$$\frac{1}{C} N^{1+s/d} \leq E_d(s, \omega_N^*) \leq C N^{1+s/d} \quad (3.2)$$

for any set of points ω_N^* minimizing $E_d(s, \omega)$ over all sets of points ω_N . Fixing N in the above and letting $s \rightarrow \infty$ then implies that for any $x_i \in \omega_N^*$, $s > d$,

$$\|x_i - x_j\| \geq C N^{-1/d}. \quad (3.3)$$

If $d = 2$, then it is a well known result of W. Habicht and B. L. van der Waerden, see [12] that

$$\delta_N = \left(\frac{8\pi}{\sqrt{3}} \right)^{1/2} N^{-1/2} + O(N^{-2/3}), \quad N \rightarrow \infty.$$

Thus for $d = 2$, the minimal energy s problem reduces to the best packing problem which is the optimal choice of points one might hope for. For $0 < s < d$, the energy integral

$$\int_{S^d} \int_{S^d} \frac{1}{\|x - y\|^s} d\sigma_d(x) d\sigma_d(y)$$

is finite and its value is

$$I_{s,d} := \frac{\Gamma((d+1)/2)\Gamma(d-s)}{\Gamma((d-s+1)/2)\Gamma(d-s/2)},$$

where Γ denotes the gamma function. Using this fact, it has been shown, see [24] and [25], that for $d-2 \leq s \leq d$

$$E_2(s, \omega_n^*) = \frac{1}{2} I_{s,d} N^2 - R_{N,s,d} \quad (3.4)$$

where

$$\frac{1}{C} \leq \frac{R_{N,s,d}}{N^{1+s/d}} \leq C. \quad (3.5)$$

In fact, for $d=2$ and $0 < s < 2$ it is conjectured, see [18], that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{R_{N,s,2}}{N^{1+s/2}} = \\ -3 \left(\frac{\sqrt{3}}{8\pi} \right)^{1/2} \zeta(1/2) \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3n+1}} - \frac{1}{\sqrt{3n+2}} \right) \approx 0.55305, \end{aligned} \quad (3.6)$$

where ζ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. In addition for $s=d$, it is known, see [16] that

$$\lim_{N \rightarrow \infty} \frac{E_d(d, \omega_N^*)}{N^2 \log N} = \frac{\Gamma((d+1)/2)}{2d\Gamma(d/2)\Gamma(1/2)}. \quad (3.7)$$

We first concentrate our efforts in computing $E_2(s, X)$, $s = 1, 2$ and 2.5 for $p = 3, \dots, 211$. The numerical data is contained in Appendix A.

Report 3.1 ($d=2$)

(a) Numerical data for $s=1$: Let

$$e_{2,3} := \frac{E_3(2, X)}{N^2}$$

and write in view of (3.4) and (3.5)

$$\frac{R_{N,1,2}}{N^2} = (1/2 - e_{1,2}) \sim \frac{1}{N^{1/2}}$$

and

$$\frac{R_{N,1,2}}{N^{3/2}} = (1/2 - e_{1,2})N^{1/2},$$

where here and throughout, for non-negative sequences a_n and b_n , $a_n \sim b_n$ if $1/C \leq a_n/b_n \leq C$.

Then for $p = 31, \dots, 211$,

$$0.004 \leq \left| \frac{R_{N,1,2}}{N^2} \right| \leq 0.008$$

and

$$1.21 \leq \left| \frac{R_{N,1,2}}{N^{3/2}} \right| \leq 1.47.$$

Thus it seems likely that $\frac{R_{N,1,2}}{N^2}$ approaches 0 and moreover we may take $C = 1.47$ in (3.5).

(b) Numerical data for $s = 2$. Let

$$e_{2,2} := \frac{E_2(2, X)}{N^2 \log N}$$

and write in view of (3.7),

$$R_{N,2,2} = |1/8 - e_{2,2}|.$$

Then for $p = 31, \dots, 211$,

$$0.025 \leq R_{N,2,2} \leq 0.205.$$

Thus $e_{2,2}$ deviates from $1/8$ with a maximum error of 0.205.

(c) Numerical data for $s = 3$. Let

$$e_{3,2} := \frac{E_2(3, X)}{N^{5/2}}.$$

Then for $p = 31, \dots, 211$,

$$0.144 \leq e_{3,2} \leq 0.259.$$

Thus we may take $C = 0.259$ in (3.2).

We next computed $E_3(s, X)$, $s = 2, 3$ and 3.125 for $p = 3, \dots, 83$. The numerical data is contained in Appendix B.

Report 3.2 ($d = 3$)

(a) Numerical data for $s = 2$: Let

$$e_{1,2} := \frac{E_2(1, X)}{N^2}$$

and write in view of (3.4) and (3.5)

$$\frac{R_{N,2,3}}{N^2} = (1/2 - e_{1,2}) \sim 1.$$

Then for $p = 31, \dots, 83$,

$$0.01 \leq \left| \frac{R_{N,2,3}}{N^2} \right| \leq 0.02$$

Thus we may take $C = 0.02$ in (3.5).

(b) Numerical data for $s = 3$. Let

$$e_{3,3} := \frac{E_3(3, X)}{N^2 \log N}$$

and write in view of (3.7),

$$R_{N,3,3} = \left| \frac{1}{3\pi} - e_{3,3} \right|.$$

Then for $p = 31, \dots, 83$,

$$0.053 \leq R_{N,3,3} \leq 0.091.$$

Thus $e_{3,3}$ deviates from $\frac{1}{3\pi}$ with a maximum error of 0.091.

(c) Numerical data for $s = 3.125$. Let

$$e_{3,3.125} := \frac{E_3(3.25, X)}{N^{2.04}}.$$

Then for $p = 31, \dots, 83$,

$$2.11 \leq e_{3,3.125} \leq 3.95.$$

Thus we may take $C = 3.95$ in (3.2).

Finally we considered numerically results on the spacings of our points for $d = 2$ and $d = 3$. To achieve our lower bounds, it suffices to apply (3.2) together with (3.1). We did this using our results above for $d = 2$, $s = 2.5$ and $d = 3$, $s = 3.125$. Our data below shows that our points on S^d , $d = 2$ and $d = 3$ are separated numerically by a factor of order $O(1/N^{1/d+1/s})$ with a perturbation factor of order $O(1/N^{1/s})$ to the ideal order $O(1/N^{1/d})$. More precisely, we have:

Report 3.3 (spacing) For any $x_i \in X$, the following hold:

(a) For $d = 2$ and $p = \{3, \dots, 211\}$

$$\|x_i - x_j\| \geq \frac{1}{(0.254)^{1/(2.5)}} \frac{1}{N^{1/2+1/(2.5)}}, i \neq j.$$

(b) For $d = 3$ and $p = \{3, \dots, 81\}$

$$\|x_i - x_j\| \geq \frac{1}{(3.95)^{1/(3.125)}} \frac{1}{N^{1/3+1/(3.125)}}, i \neq j.$$

4 Extension to finite fields of odd prime powers

In this last section, we briefly observe that we may solve the same quadratic form (1.1) over a general finite field F_q , where $q = p^e$ is an odd prime power and in this way distribute points on S^d as well. One way to do this is as follows. Assume that $q = p^e$, with $e \geq 1$. Then the field F_q is an e -dimensional vector space over the field F_p . Let $\alpha_1, \dots, \alpha_e$ be a basis of F_q over F_p . Thus if $\alpha \in F_q$, then α can be uniquely written as $\alpha = a_1\alpha_1 + \dots + a_e\alpha_e$, where each $a_i \in F_p$. Moreover, we may assume that each a_i satisfies $-(p-1)/2 \leq a_i \leq (p-1)/2$.

If (x_1, \dots, x_{d+1}) is a solution to the quadratic form (1.1) over F_q , then each x_i is of the form $x_i = \alpha \in F_q$. Corresponding to the finite field element $x_i = \alpha$, we may now naturally associate the integer $M_i = a_1 + a_2p + \dots + a_ep^{e-1}$. It is an easy exercise to check that indeed $-(p^e-1)/2 \leq M_i \leq (p^e-1)/2$. We then map the vector $V = (M_1, \dots, M_{d+1})$ to the surface of the unit sphere S^d by normalizing the vector V . We note that when $e = 1$, this reduces to our original construction. In particular, for increasing values of e , we obtain an increasing number N_e of points scattered on the surface of the unit sphere S^d , so that as $e \rightarrow \infty$, it is clear that $N_e \rightarrow \infty$.

In light of the success of our initial investigation for the special case $e = 1$, a detailed analysis of the cases where $e > 1$ may warrant further investigation.

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Appendix A, Table of normalized $E_2(s, X)$, $s = 1, 2, 2.5$
For convenience, we adopt the following conventions:

Set $e1 := e_{1,2} = \frac{E_2(1,X)}{N^2}$, $e2 := e_{2,2} = \frac{E_2(2,X)}{N^2 \log N}$, $e3 := e_{3,2} := \frac{E_2(3,X)}{N^{5/2}}$.

$p = 3$	$e1, e2, e3 =$	0.27736892706218	0.01046457424783	0.11980183204618
$p = 5$	$e1, e2, e3 =$	0.40195539994316	0.11628809346709	0.16900002651468
$p = 7$	$e1, e2, e3 =$	0.43294284766539	0.14155651348989	0.20783488490901
$p = 11$	$e1, e2, e3 =$	0.47263289296172	0.15758965144183	0.22872928195394
$p = 13$	$e1, e2, e3 =$	0.46734388392531	0.13930424776208	0.19737146213200
$p = 17$	$e1, e2, e3 =$	0.47831260388752	0.14292677178733	0.19559212838090
$p = 19$	$e1, e2, e3 =$	0.48862305761514	0.15647575327531	0.21230845889239
$p = 23$	$e1, e2, e3 =$	0.49680020536376	0.16083024644030	0.21114298075140
$p = 29$	$e1, e2, e3 =$	0.49963999603979	0.20411306813044	0.25438001898121
$p = 31$	$e1, e2, e3 =$	0.50558991175900	0.20872789826935	0.25835091872858
$p = 37$	$e1, e2, e3 =$	0.49679590269129	0.16908367785692	0.20014914672875
$p = 41$	$e1, e2, e3 =$	0.49967248912416	0.18213470230307	0.21067502961875
$p = 43$	$e1, e2, e3 =$	0.50082510107625	0.15645393314738	0.17997113615146
$p = 47$	$e1, e2, e3 =$	0.50395991784508	0.17448493068513	0.19648831884456
$p = 53$	$e1, e2, e3 =$	0.50302988221990	0.20628228220202	0.22447505302842
$p = 59$	$e1, e2, e3 =$	0.50770676508130	0.21250313388376	0.22610594444172
$p = 61$	$e1, e2, e3 =$	0.50495658406683	0.22070120695148	0.23184436755631
$p = 67$	$e1, e2, e3 =$	0.50654515261995	0.19456180151876	0.20028103675876
$p = 71$	$e1, e2, e3 =$	0.50799678645242	0.19531553592815	0.19798724786361
$p = 73$	$e1, e2, e3 =$	0.50138319997937	0.17763041043367	0.17807400648780
$p = 79$	$e1, e2, e3 =$	0.50667682389347	0.24131448167150	0.23767087684753
$p = 83$	$e1, e2, e3 =$	0.50632565820368	0.19072462271388	0.18532160079489
$p = 89$	$e1, e2, e3 =$	0.50420397664596	0.19014449465661	0.18065926631423
$p = 97$	$e1, e2, e3 =$	0.50484339249495	0.17456698846262	0.16193580914486
$p = 101$	$e1, e2, e3 =$	0.50561387064791	0.21342480676679	0.19574508602563
$p = 103$	$e1, e2, e3 =$	0.50800632405351	0.19108133195127	0.17476558366460
$p = 107$	$e1, e2, e3 =$	0.50744167603236	0.21193413555663	0.19173518130406
$p = 109$	$e1, e2, e3 =$	0.50530215087720	0.21451186060578	0.19252922201869
$p = 113$	$e1, e2, e3 =$	0.50508191824201	0.22699743097426	0.20164193844709
$p = 127$	$e1, e2, e3 =$	0.50685489076967	0.18562274384222	0.15976587408138
$p = 131$	$e1, e2, e3 =$	0.50784482237133	0.21884299920671	0.18664215587238
$p = 137$	$e1, e2, e3 =$	0.50588778915797	0.22528035319823	0.18918543160007
$p = 139$	$e1, e2, e3 =$	0.50868464653880	0.22665247011204	0.18992843422136
$p = 149$	$e1, e2, e3 =$	0.50654747356646	0.25299360855091	0.20721585270831
$p = 151$	$e1, e2, e3 =$	0.50839219613404	0.21335681640995	0.17440146172139
$p = 157$	$e1, e2, e3 =$	0.50550126525598	0.24862960416529	0.20046770502092
$p = 163$	$e1, e2, e3 =$	0.50708965412530	0.18736597295888	0.14964778305454
$p = 167$	$e1, e2, e3 =$	0.50726676470783	0.19666444095315	0.15591738945230
$p = 173$	$e1, e2, e3 =$	0.50594513571582	0.20098724096747	0.15735377427577
$p = 179$	$e1, e2, e3 =$	0.50824611328744	0.33454655513670	0.25964637978174
$p = 181$	$e1, e2, e3 =$	0.50687976341157	0.27359266111448	0.21125394734360
$p = 191$	$e1, e2, e3 =$	0.50841113729910	0.20611531648543	0.15679195137179
$p = 193$	$e1, e2, e3 =$	0.50641535139461	0.23733139042064	0.17966621151199
$p = 197$	$e1, e2, e3 =$	0.50665988263156	0.22671894926843	0.17054554488575
$p = 199$	$e1, e2, e3 =$	0.50721503705415	0.22942421796768	0.17230976048709
$p = 211$	$e1, e2, e3 =$	0.50723259692356	0.19527807424465	0.14400243632793

Appendix B, Table of normalized $E_3(s, X)$, $s = 2, 3, 3.125$
For convenience, we adopt the following conventions:

$$\text{Set } e1 := e_{2,3} = \frac{E_3(2,X)}{N^2}, e2 := e_{3,3} = \frac{E_3(3,X)}{N^2 \log N}, e3 := e_{3,3.125} := \frac{E_3(3.125,X)}{N^{2.04}}.$$

$p = 3$	$e1, e2, e3 =$	0.28993055555556	0.07726112747611	0.24132173335499
$p = 5$	$e1, e2, e3 =$	0.37650018037519	0.08604648073508	0.42113832739910
$p = 7$	$e1, e2, e3 =$	0.42533543875492	0.09834709791994	0.60593660203289
$p = 11$	$e1, e2, e3 =$	0.49141162848846	0.14167115959202	1.1735995123910
$p = 13$	$e1, e2, e3 =$	0.48120035766978	0.12590020472700	1.1139764584208
$p = 17$	$e1, e2, e3 =$	0.52132037047210	0.19355085551622	2.0982770508338
$p = 19$	$e1, e2, e3 =$	0.50870874907126	0.14945260382314	1.6099778963136
$p = 23$	$e1, e2, e3 =$	0.50664591863877	0.15354023703434	1.8296037700500
$p = 29$	$e1, e2, e3 =$	0.51703948372378	0.15825300052952	2.0440071303765
$p = 31$	$e1, e2, e3 =$	0.51738403568925	0.15916399708017	2.1127707717489
$p = 37$	$e1, e2, e3 =$	0.51884012094570	0.18738716021182	2.8402487195221
$p = 41$	$e1, e2, e3 =$	0.52328941444401	0.17241718697984	2.5921542948948
$p = 43$	$e1, e2, e3 =$	0.52006281540841	0.16742067897271	2.5555792369681
$p = 47$	$e1, e2, e3 =$	0.52154686231626	0.19360006853089	3.2832846733182
$p = 53$	$e1, e2, e3 =$	0.52302701242501	0.21637834813051	3.9535044537455
$p = 59$	$e1, e2, e3 =$	0.52414896518911	0.26040121306624	5.3775580289229
$p = 67$	$e1, e2, e3 =$	0.52352555470433	0.16356808691818	2.8640336098518
$p = 71$	$e1, e2, e3 =$	0.52278118326934	0.16964255435064	3.0594908965203
$p = 73$	$e1, e2, e3 =$	0.52466921979023	0.19727729124473	3.8343687834560
$p = 79$	$e1, e2, e3 =$	0.52456644056252	0.18994184763923	3.7615796423715
$p = 83$	$e1, e2, e3 =$	0.52440012994288	0.18298261836839	3.5924738028014